# Curvature, gauge transformations and the Bianchi identity

Simone Haupt

05-11-2026

Seminar on Gauge Theory – Talk 3

WiSe2026/2027

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# Definition of curvature - I

- For ordinary scalar function $f$, with definition of Lie bracket
$$X(Yf) - Y(Xf) - [X,Y]f = 0 \Leftrightarrow [X,Y] = XY - YX$$
  - This does **not** hold for connections, instead of 0 we get the curvature

**Definition 3.1 (Curvature of a vector connection)**

For vector fields $X, Y$ and a section $s$ of $E$, we define:
$$R^{\nabla}(X,Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s$$

**Proposition 3.2 (Curvature is an endomorphism-valued two-form)**

$R^{\nabla}(X,Y)s$ is function-linear in $X$, $Y$ and $s$. It is also skew-symmetric in $X, Y$. Therefore
$$R^{\nabla}(X,Y)s \in \Omega^2\big(M; \mathrm{End}(E)\big)$$

# Definition of curvature - II

**Proof - I**

- Skew-symmetry follows directly from the definition and antisymmetry of the Lie bracket
  - $R^{\nabla}(X,Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \underbrace{\nabla_{[X,Y]} s}_{-\nabla_{[Y,X]} s} = -R^{\nabla}(Y,X)s$
- For function-linearity in the first direction we use

$$[fX,Y] = f[X,Y] - Y(f)X$$

  - $R^{\nabla}(fX,Y)s = f\nabla_X \nabla_Y s - \nabla_Y(f\nabla_X s) - f\nabla_{[X,Y]} s + Y(f)\nabla_X s = fR^{\nabla}(X,Y)s$
- Second direction follows with skew-symmetry

# Definition of curvature - III

**Proof - II**

- For function-linearity in a section we look at
  - $R^{\nabla}(X,Y)(us) = \nabla_X \nabla_Y (us) - \nabla_Y \nabla_X (us) - \nabla_{[X,Y]}(us)$
    - $\nabla_X \nabla_Y (us) = X\big(Y(u)\big)s + Y(u)\nabla_X s + X(u)\nabla_Y s + u\nabla_X \nabla_Y s$
    - $\nabla_Y \nabla_X (us) = Y\big(X(u)\big)s + X(u)\nabla_Y s + Y(u)\nabla_X s + u\nabla_Y \nabla_X s$
    - $\nabla_{[X,Y]}(us) = [X,Y](u)s + u\nabla_{[X,Y]} s$
  - $R^{\nabla}(X,Y)(us) = u\nabla_X \nabla_Y s - u\nabla_Y \nabla_X s - u\nabla_{[X,Y]} s = uR^{\nabla}(X,Y)s$

□

- Curvature is first-order data about a connection, but acts pointwise on a section

# Multiplying matrix-valued differential forms - I

For $\alpha, \beta$ matrices with entries forms of degrees $p$ and $q$ respectively we define

$$(\alpha \wedge \beta)^a_b = \sum_c \alpha^a_c \wedge \beta^c_b$$

- associative
- $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$
- not generally graded commutative
  - Example: $A = Bdx + Cdy$ then

$$A \wedge A = \overbrace{(BC - CB)}^{[B,C]} dx \wedge dy$$

- individual scalar one-forms anticommute
- for line bundle coefficients are scalars and $A \wedge A = 0$
  - not for every matrix of one-forms

# Multiplying matrix-valued differential forms - II

- Graded bracket is

$$[\alpha, \beta] = \alpha \wedge \beta - (-1)^{pq} \beta \wedge \alpha$$

  - For two one-forms $[A, A] = 2A \wedge A$
    - $[A, A] = A \wedge A - (-1)^1 A \wedge A = A \wedge A + A \wedge A = 2(A \wedge A)$
  - Bracket has value in Lie algebra
    - write forms in a Lie-algebra basis
    - coefficients are brackets of the basis matrices
    - independent of a chosen embedding of the Lie algebra into matrices
- Trace has grades cyclic property

$$\mathrm{tr}(\alpha \wedge \beta) = (-1)^{pq} \mathrm{tr}(\beta \wedge \alpha)$$

  - does not mean that the two matrix-valued forms are equal

# Local formula for curvature - I

- In a frame write $\nabla = d + A$ and $A = \sum_i A_i dx^i$
- Compute operator commutator on coefficient column

$$\left[\partial_i + A_i, \partial_j + A_j\right] = \partial_i A_j - \partial_j A_i + A_i A_j - A_j A_i$$

  - coordinate partial derivatives commute → second derivatives cancel
  - product rule → first derivatives of the column cancel
- Define local curvature matrix

$$F_A = dA + A \wedge A = dA + \frac{1}{2}[A, A]$$

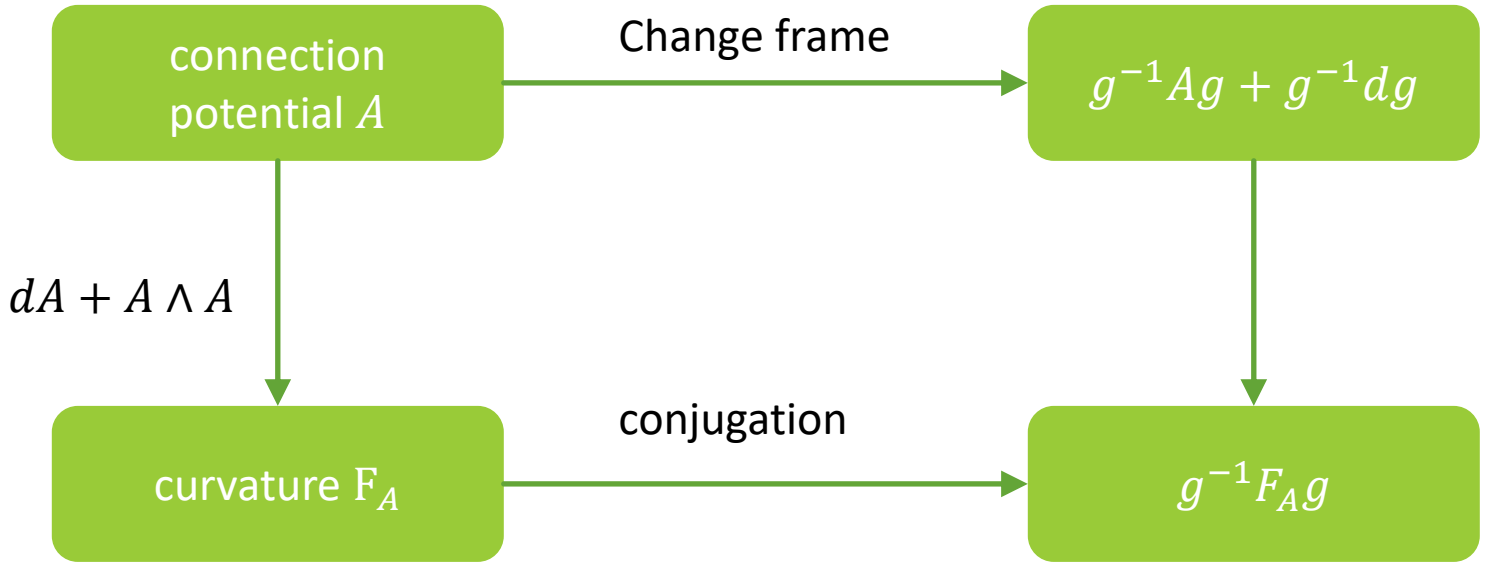
  - Then $R^{\nabla}(X, Y)(ev) = e(F_A(X, Y)v)$
    - In coordinates

$$F_{ij} = \partial_i A_j - \partial_j A_i + \left[A_i, A_j\right]$$

$$F_A = \sum_{i<j} F_{ij} dx^i \wedge dx^j = \frac{1}{2} \sum_{i,j} F_{ij} dx^i \wedge dx^j$$

# Local formula for curvature - II

- When changing frame to $e' = eg$ we get $F_{A'} = g^{-1}F_A g$ because $R^{\nabla}$ is an endomorphism-valued form




  - No term $g^{-1}dg$ in the lower right corner since curvature is tensorial
- Diagram commutes, since we get the same end result

# Local formula for curvature - III

**Definition 3.3 (Flat connection)**

A connection is flat if its curvature is zero. $F_A = 0$

This is a condition on the connection, not on the underlying bundle.

# Curvature on a principal bundle - I

For principal connection form $\omega \in \Omega^1(P;\mathfrak{g})$ we define a form on $P$:

- $\Omega = d\omega + \frac{1}{2}[\omega,\omega] = d\omega + \omega \wedge \omega$
  - additional property: horizontality

**Proposition 3.4 (Horizontality and equivariance)**

The principal curvature satisfies horizontality

$$\Omega(\xi^{\#}, Z) = 0$$

and equivariance

$$R_g^* \Omega = \mathrm{Ad}_{g^{-1}} \Omega$$

It therefore defines a two-form on $M$ with values in $\mathrm{ad}(P)$.
In a local section $\sigma$, that form has matrix $\sigma^* \Omega = dA + A \wedge A$.

# Curvature on a principal bundle - II

**Proof - I**

$\Omega(\xi^{\#}, Z) = 0$

*Horizontality*

- flow of $\xi^{\#}$ is $R_{e^{t\xi}}$
  - $\xi^{\#}$ fundamental vertical vector field
- Differentiating connection equivariance $R_g^* \omega = \mathrm{Ad}_{g^{-1}} \omega$ gives
$$\mathcal{L}_{\xi^{\#}} \omega = -[\xi, \omega]$$
- For any vector field *Z*, the definition of the exterior derivative of a one-form gives
$$d\omega(\xi^{\#}, Z) = \xi^{\#}(\omega(Z)) - Z\left(\omega(\xi^{\#})\right) - \omega([\xi^{\#}, Z]) = (\mathcal{L}_{\xi^{\#}}\omega)(Z) = -[\xi, \omega(Z)]$$
because $\omega(\xi^{\#}) = \xi$ is constant.
- Also $(\omega \wedge \omega)(\xi^{\#}, Z) = [\xi, \omega(Z)]$
- Adding both we get
$$\Omega(\xi^{\#}, Z) = d\omega(\xi^{\#}, Z) + (\omega \wedge \omega)(\xi^{\#}, Z) = -[\xi, \omega(Z)] + [\xi, \omega(Z)] = 0$$

# Curvature on a principal bundle - III

**Proof - II**

$R_g^*\Omega = \mathrm{Ad}_{g^{-1}}\Omega$

*Equivariance*

- Pulling back $\Omega = d\omega + \omega \wedge \omega$ by $R_g$
  - $R_g^*\Omega = R_g^*(d\omega) + R_g^*(\omega \wedge \omega)$
- Pullback commutes with exterior differentiation and preserves wedge products and brackets
  - $R_g^*\Omega = d\left(R_g^*\omega\right) + \left(R_g^*\omega\right) \wedge \left(R_g^*\omega\right)$
- With connection equivariance: $R_g^*\omega = \mathrm{Ad}_{g^{-1}}\omega$
- $\mathrm{Ad}_{g^{-1}}$ is a constant linear map that commutes with the exterior derivative and preserves the bracket
  - $R_g^*\Omega = d\left(\mathrm{Ad}_{g^{-1}}\omega\right) + \left(\mathrm{Ad}_{g^{-1}}\omega\right) \wedge \left(\mathrm{Ad}_{g^{-1}}\omega\right) = \mathrm{Ad}_{g^{-1}}(d\omega + \omega \wedge \omega) = \mathrm{Ad}_{g^{-1}}\Omega$

# Curvature on a principal bundle - IV

**Proof - III**

two-form on $M$
values in $\mathrm{ad}(P)$
$\sigma^*\Omega = dA + A \wedge A$

For $X, Y \in T_xM$, choose $p \in P_x$ and any lifts $\tilde{X}, \tilde{Y} \in T_pP$.

- the adjoint-bundle-valued form is defined by class $\left[p, \Omega_p\left(\tilde{X}, \tilde{Y}\right)\right]$ which is
  - independent of the lifts by horizontality
  - independent of $p$ by equivariance and the associated-bundle relation
- pullback by $\sigma$ commutes with $d$ and the wedge product giving

$$\sigma^*\Omega = dA + A \wedge A$$

□

# Curvature on a principal bundle - V

- For horizontal lifts of vector fields, the structure equation simplifies to

$$\Omega(X^H, Y^H) = -\omega([X^H, Y^H])$$

  - both one-form values $\omega(X^H)$, $\omega(Y^H)$ vanish, leaving only the bracket term in $d\omega$

- curvature measures the vertical part of the failure of horizontal vector fields to commute within the horizontal distribution

  - principal meaning of the small-loop effect

- If $E = P \times_\rho V$, the curvature of its induced connection is $\rho_*(\Omega)$, interpreted as an endomorphism-valued form on the base

# Covariant exterior differentiation - I

Write $\Omega^p(M;E) = \Gamma(\Lambda^p T^*M \otimes E)$, complexifying real forms when appropriate.

**Proposition 3.5 (The extension to forms)**

There is a unique operator $d_\nabla : \Omega^p(M;E) \to \Omega^{p+1}(M;E)$ equal to $\nabla$ in degree zero and for $\alpha \in \Omega^p(M)$ satisfying

$$d_\nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^p \alpha \wedge \nabla s$$

In a frame, on a column $\psi$ of $p$-forms it is

$$d_\nabla \psi = d\psi + A \wedge \psi$$

# Covariant exterior differentiation - II

**Proof**

- Define the operator locally by
$$d_\nabla \psi = d\psi + A \wedge \psi$$
- If $\psi' = g^{-1}\psi$ and $A' = g^{-1}Ag + g^{-1}dg$, the product rule and $dg^{-1} = -g^{-1}(dg)g^{-1}$ give
$$d\psi' + A' \wedge \psi' = g^{-1}(d\psi + A \wedge \psi)$$
- Expanding a simple tensor and moving the degree-one form $A$ past $\alpha$ gives the sign $(-1)^p$ in
$$d_\nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^p \alpha \wedge \nabla s$$
- Every bundle-valued form is locally a sum of such simple tensors in a frame
  - the rule proves uniqueness.
- respects $f\alpha \otimes s = \alpha \otimes fs$
  - two product rules both give $df \wedge \alpha \otimes s$ as the extra term

□

# Covariant exterior differentiation - III

For endomorphism-valued $p$-forms, the induced connection has the different local expression

$$d_A\eta = d\eta + [A,\eta] = d\eta + A\wedge\eta - (-1)^p\eta\wedge A$$

- degree zero

$$dT + AT - TA$$

- additional right-hand term reflects the action on both the output and input of an endomorphism.
- The same formula defines covariant exterior differentiation on adjoint-bundle-valued forms.

**Proposition 3.6 (The square is curvature)**

For an $E$-valued form $\psi$ and an endomorphism-valued form $\eta$ is

$$d_\nabla^2\psi = F_A\wedge\psi, \qquad d_A^2\eta = [F_A,\eta]$$

# Covariant exterior differentiation - IV

**Proof**

- Expanding gives
$$d_\nabla^2\psi = (d + A\wedge)^2\psi = dA\wedge\psi - A\wedge d\psi + A\wedge d\psi + A\wedge A\wedge\psi = F_A\wedge\psi$$
- Expanding gives
$$d_A^2\eta = d[A,\eta] + [A,d\eta] + \big[A,[A,\eta]\big]$$
- For an endomorphism-valued *p*-form, the ordinary product rule gives
$$d[A,\eta] = [dA,\eta] - [A,d\eta]$$
- Expanding graded bracket twice gives
$$\big[A,[A,\eta]\big] = A\wedge A\wedge\eta - (-1)^p A\wedge\eta\wedge A - (-1)^{p+1}A\wedge\eta\wedge A + (-1)^{2p+1}\eta\wedge A\wedge A$$
$$= [A\wedge A,\eta]$$
- It follows
$$d_A^2\eta = [dA,\eta] - [A,d\eta] + [A,d\eta] + [A\wedge A,\eta] = [dA + A\wedge A,\eta] = [F_A,\eta]$$

□

# Covariant exterior differentiation - V

- ordinary exterior derivative always squares to zero
  - covariant extension generally does not
- applying the adjoint formula to the matrix $A$ gives
$$d_A A = dA + [A, A] = dA + 2A \wedge A$$
not $F_A$
- structure equation $F_A = dA + A \wedge A$ for curvature
- $d_A$ on tensorial adjoint-bundle-valued forms

# Gauge transformations - I

**Definition 3.7 (Gauge transformation)**

A gauge transformation of a vector bundle is a smooth fibrewise linear automorphism $\Phi : E \to E$ covering $\mathrm{id}_M$, preserving any specified metric and determinant. It acts on connections by

$$(\Phi^* \nabla)_X s = \Phi^{-1}\big(\nabla_X(\Phi s)\big)$$

For a principal bundle it is a smooth $G$-equivariant automorphism $\varphi : P \to P$ covering $\mathrm{id}_M$, acting by pullback of the connection form. The set of these maps is the gauge group.

# Gauge transformations - II

- In a fixed vector frame write $\phi(ev) = euv$. Substitution gives

$$\phi^*\nabla = d + A^u, \qquad A^u = u^{-1}Au + u^{-1}du$$

  - Transformation of a Connection
  - $u(x) \in U(r)$, if $\Phi$ metric-preserving in a unit frame
  - $u(x) \in SU(r)$, if $\Phi$ also preserves fixed determinant
  - For a nontrivial bundle local matrices $u$ are not one global matrix-valued function on $M$
  - If $e_j = e_i g_{ij}$, the condition that they represent the same $\phi$ is

$$u_j = g_{ij}^{-1} u_i g_{ij}$$

# Gauge transformations - III

- For a principal bundle there is an equivalent global description

$$\varphi(p) = pu(p), \qquad u(pg) = g^{-1}u(p)g$$

  - uniquely determined by freeness of the action
  - equivariance of $\varphi \Leftrightarrow u(pg) = g^{-1}u(p)g$
  - local values are $u_i = u \circ \sigma_i$
  - local pullback formula for a principal connection is

$$\phi^*\nabla = d + A^u, \qquad A^u = u^{-1}Au + u^{-1}du$$

  - For an abelian group the conjugation condition disappears, so $u$ is constant along fibres and is a function on the base

# Curvature and transport under a gauge transformation - I

**Proposition 3.6.2 (Gauge covariance)**

For the convention

$$\phi^*\nabla = d + A^u, \qquad A^u = u^{-1}Au + u^{-1}du$$

is

$$F_{A^u} = u^{-1}F_A u, \qquad d_{A^u}(u^{-1}\eta u) = u^{-1}(d_A\eta)u$$

For vector-valued forms the corresponding formula is $d_{\nabla^u}(u^{-1}\psi) = u^{-1}d_\nabla\psi$.

Parallel transport transforms at the endpoints by

$$P_\gamma^{\nabla^u} = \Phi_{\gamma(b)}^{-1} P_\gamma^\nabla \Phi_{\gamma(a)}$$

# Curvature and transport under a gauge transformation - II

$$F_{A^u} = u^{-1} F_A u$$

**Proof - I**

- Set $B = u^{-1}Au$ and $\vartheta = u^{-1}du$
  - $A^u = B + \vartheta$
- Differentiating an inverse gives $d\vartheta = -\vartheta \wedge \vartheta$
- Product rule gives $dB = u^{-1}(dA)u - \vartheta \wedge B - B \wedge \vartheta$
- Computing $F_{A^u}$ gives

$$F_{A^u} = dB + d\vartheta + B \wedge B + B \wedge \vartheta + \vartheta \wedge B + \vartheta \wedge \vartheta$$
$$= u^{-1}(dA)u - \vartheta \wedge B - B \wedge \vartheta - \vartheta \wedge \vartheta + B \wedge B + B \wedge \vartheta + \vartheta \wedge B + \vartheta \wedge \vartheta$$
$$= u^{-1}(dA)u + B \wedge B$$

  - $B \wedge B = u^{-1}(A \wedge A)u$
  - $F_{A^u} = u^{-1}(dA)u + u^{-1}(A \wedge A)u = u^{-1}F_A u$

# Curvature and transport under a gauge transformation - III

$$d_{A^u}(u^{-1}\eta u) = u^{-1}(d_A\eta)u$$

**Proof - II**

- Be $\eta$ endomorphism-valued differential form
  - $d_{A^u}(u^{-1}\eta u) = d(u^{-1}\eta u) + A^u \wedge (u^{-1}\eta u) - (-1)^p(u^{-1}\eta u) \wedge A^u$
  - $d(u^{-1}\eta u) = -\vartheta \wedge (u^{-1}\eta u) + u^{-1}(d\eta)u + (-1)^p(u^{-1}\eta u) \wedge \vartheta$
- $d_{A^u}(u^{-1}\eta u) = -\vartheta \wedge (u^{-1}\eta u) + u^{-1}(d\eta)u + (-1)^p(u^{-1}\eta u) \wedge \vartheta + (u^{-1}Au + \vartheta) \wedge (u^{-1}\eta u) - (-1)^p(u^{-1}\eta u) \wedge (u^{-1}Au + \vartheta)$

$$= u^{-1}(d\eta)u + u^{-1}Au \wedge u^{-1}\eta u - (-1)^p u^{-1}\eta u \wedge u^{-1}Au$$
$$= u^{-1}(d\eta)u + u^{-1}(A \wedge \eta)u - (-1)^p u^{-1}(\eta \wedge A)u$$
$$= u^{-1}(d_A\eta)u$$

# Curvature and transport under a gauge transformation - IV

$$P_\gamma^{\nabla^u} = \Phi_{\gamma(b)}^{-1} P_\gamma^\nabla \Phi_{\gamma(a)}$$

**Proof - III**

- a section $V$ along $\gamma$ is parallel for $\nabla^u$ exactly when $\Phi V$ is parallel for $\nabla$
  - $V(b) = P_\gamma^{\nabla^u} V(a)$
  - $(\Phi V)(b) = P_\gamma^\nabla (\Phi V)(a) \Leftrightarrow \Phi_{\gamma(b)} V(b) = P_\gamma^\nabla \Phi_{\gamma(a)} V(a)$
    $\Leftrightarrow V(b) = \Phi_{\gamma(b)}^{-1} P_\gamma^\nabla \Phi_{\gamma(a)} V(a)$

□

# Curvature and transport under a gauge transformation - V

| Operation | Changes | Fixed |
|---|---|---|
| Frame change | Local section or basis | Geometric connection |
| Gauge action | Connection pulled back by bundle symmetry | Local frame |

- Both give same matrix expression
  - applying a bundle map to a chosen frame gives another frame
  - either object or description is changing

# Infinitesimal gauge transformations

Let $\xi$ be a local Lie-algebra-valued function representing a section of $\mathrm{ad}(P)$, and let $u_t = e^{t\xi}$. Differentiating at $t = 0$ gives

$$\frac{d}{dt}\Big|_0 A^{u_t} = d\xi + [A, \xi] = d_A \xi, \qquad \frac{d}{dt}\Big|_0 F_A{}^{u_t} = [F_A, \xi]$$

Thus $d_A$ on zero-forms describes the infinitesimal directions in which a gauge symmetry moves a connection.

# The Bianchi identity - I

**Theorem 3.7.1 (Bianchi identity)**

Every connection satisfies

$$d_A F_A = dF_A + A \wedge F_A - F_A \wedge A = 0$$

For a principal connection form, the corresponding total-space identity is

$$d\Omega + \omega \wedge \Omega - \Omega \wedge \omega = 0$$

- Bianchi identity in coordinates

$$D_i F_{ik} + D_j F_{ki} + D_k F_{ij} = 0, \qquad D_i T = \partial_i T + [A_i, T]$$

  - $F_{ij}$ components of the curvature
  - $D_i$ gauge-covariant derivative

# The Bianchi identity - II

$$d_A F_A = dF_A + A \wedge F_A - F_A \wedge A = 0$$
$$d\Omega + \omega \wedge \Omega - \Omega \wedge \omega = 0$$

**Proof**

- differentiating using $d^2 = 0$
  - $dF_A = dA \wedge A - A \wedge dA$
  - $A \wedge F_A - F_A \wedge A = A \wedge dA + A \wedge A \wedge A - dA \wedge A - A \wedge A \wedge A$
- Substituting gets $d_A F_A = 0$
- For the other equation we replace $A \to \omega$ and $F_A \to \Omega$
- Pulling the principal identity back along a section gives back the first equation

□

# Examples - I - large potential with zero curvature

- On a trivial unitary line over a coordinate ball
- Zero connection $u_N(x) = e^{iNx^1}$
- Gauge-Transform $A_N = u_N^{-1} du_N = iNx^1$
- curvature is zero for every $N$

$$F_{A_N} = dA_N + A_N \wedge A_N$$
$$= d(iNdx^1) = iNd(dx^1) = 0$$

- size of the potential grows with $N$

- curvature bounds alone cannot bound the coefficients in every gauge
- Gauge fixing is essential in compactness theorems.

# Examples - II - nonabelian Bianchi cancellation

- On $\mathbb{R}^3$ set $A = x\tau_1 dy + \tau_2 dz$
- $F_A = dA + A \wedge A = \tau_1 dx \wedge dy + 2x\tau_3 dy \wedge dz$
- Bianchi identity

$$d_A F_A = dF_A + A \wedge F_A - F_A \wedge A = 0$$

  - $dF_A = 2\tau_3 dx \wedge dy \wedge dz$
- $A \wedge F_A - F_A \wedge A$
$= (x\tau_1 dy + \tau_2 dz) \wedge (\tau_1 dxdy + 2x\tau_3 dydz) - (\tau_1 dxdy + 2x\tau_3 dydz) \wedge (x\tau_1 dy + \tau_2 dz)$
$= \tau_2\tau_1 dz \wedge dx \wedge dy - \tau_1\tau_2 dx \wedge dy \wedge dz$
$= \overbrace{(\tau_2\tau_1 - \tau_1\tau_2)}^{-[\tau_1,\tau_2]=-2\tau_3} dx \wedge dy \wedge dz$
$= -2\tau_3 dx \wedge dy \wedge dz$
- Bianchi identity in coordinates

$$D_x F_{yz} + D_y F_{zx} + D_z F_{xy} = 0$$

  - $D_x F_{yz} = 2\tau_3$
  - $D_y F_{zx} = 0$
  - $D_z F_{xy} = -2\tau_3$

# Important Formulas

- $F_A = dA + A \wedge A$
- $F_{A^u} = u^{-1} F_A u$
- $d_\nabla \psi = d\psi + A \wedge \psi$
- $d_A \eta = d\eta + [A, \eta]$
- $d_\nabla^2 \psi = F_A \wedge \psi$
- $d_A^2 \eta = [F_A, \eta]$
- $d_A F_A = 0$